

NEW EXAMPLES OF TERMINAL AND LOG CANONICAL SINGULARITIES

JÁNOS KOLLÁR

The aim of this note is to revisit the constructions of [Rei80, FM83, Rei94] to obtain new examples of terminal and log canonical singularities. First we discuss the general method and then work out in detail the following series of examples.

Theorem 1. *For every $r \geq 4$ there are germs of terminal 4-fold singularities $(0 \in X_r)$ such that K_{X_r} is Cartier, $X_r \setminus \{0\}$ is simply connected, the class group $\text{Cl}(X_r)$ is trivial and the embedding dimension of X_r is r .*

Theorem 2. *Let F be a connected 2-manifold without boundary. Then there are germs of isolated log canonical 3-fold singularities $(0 \in X = X(F))$ such that for any resolution $p : Y \rightarrow X$ we have $R^1p_*\mathcal{O}_Y \cong H^1(F, \mathbb{C})$, $\pi_1(Y) \cong \pi_1(F)$ and $\pi_1(X \setminus \{0\})$ is an extension of $\pi_1(F)$ by a cyclic group.*

3 (Previous examples). One of the first results on the singularities of higher dimensional birational geometry, proved in [Rei80], says that a 3-fold terminal singularity whose canonical class is Cartier is a hypersurface double point. Thus every 3-fold terminal singularity is a quotient of a hypersurface double point by a cyclic group action. This description can be developed into a complete list; see [Rei87].

For some time it has been an open question if, in some sense, terminal singularities in higher dimensions also form an essentially bounded family. The first relevant examples are in an unpublished note [BR10].

In earlier examples of germs of isolated log canonical 3-fold singularities $(0 \in X)$ with a resolution $p : Y \rightarrow X$ we had $\dim R^1p_*\mathcal{O}_Y \leq 2$ (with equality only for cones over an Abelian surface) and $\pi_1(Y)$ contained a finite index Abelian subgroup.

In previous higher dimensional examples we had $\dim R^1p_*\mathcal{O}_Y \leq \dim X - 1$ and $\pi_1(Y)$ contained a finite index Abelian subgroup. These both hold for cones over smooth varieties.

It is also worthwhile to emphasize the difference between the log canonical and the dlt cases. Let $(0 \in X, \Delta)$ be a germ of a dlt pair and $p : Y \rightarrow X$ any resolution of singularities. Then $R^ip_*\mathcal{O}_Y = 0$ for $i > 0$ by [Elk81] (a simple proof is in [KM98, 5.22]) and $\pi_1(Y) = 1$ by [Kol93, Tak03].

Singularities with prescribed exceptional divisors.

Every smooth variety Z can be realized as the exceptional set of a resolution of an isolated singular point $(0 \in X)$; one can simply take X to be a cone over Z . More generally, given any scheme Z of dimension n , one can ask if there is a normal, isolated singularity $0 \in X$ of dimension $n + 1$ with a resolution

$$\begin{array}{ccc} Z & \subset & Y \\ \downarrow & & \downarrow \\ 0 & \in & X \end{array} \quad \text{such that } Y \setminus Z \cong X \setminus \{0\}.$$

An obvious restriction is that Z should have only hypersurface singularities, but this is not sufficient. For instance, consider $Z = (xy = 0) \subset \mathbb{P}^3$ and let Y be any smooth 3-fold containing Z . Then the normal bundle of the line $L := (x = y = 0) \subset Z$ in Y is $\mathcal{O}_{\mathbb{P}^1}(1) + \mathcal{O}_{\mathbb{P}^1}(1)$, thus $L \subset Y$ deforms in every direction. Hence Z is not contractible; it can not even be a subscheme of the exceptional set of a resolution of an isolated 3-fold singularity.

It turns out, however, that if we allow Y to be (very mildly) singular, then one can construct such $(0 \in X)$.

Proposition 4. *Let P be a smooth variety and $Z \subset P$ a subscheme of codimension n that is a local complete intersection. Let L be a line bundle on P such that $L(-Z)$ is generated by global sections. Let $Z \subset Y \subset P$ be the complete intersection of $(n-1)$ general sections of $L(-Z)$. Set*

$$\pi : B_{(-Z)}Y := \text{Proj}_Y \sum_{m=0}^{\infty} \mathcal{O}_Y(mZ) \rightarrow Y.$$

(We blow up not the ideal sheaf of Z but its inverse in the class group.) Then

- (1) $B_{(-Z)}Y$ is CM.
- (2) $\pi_*^{-1}Z$ is isomorphic to Z and it is a Cartier divisor in $B_{(-Z)}Y$.
- (3) The exceptional set of π is a \mathbb{P}^1 -bundle of codimension 2 in $B_{(-Z)}Y$.
- (4) If Z has only normal crossing singularities then $B_{(-Z)}Y$ is terminal in a neighborhood of Z .
- (5) If Z has only normal crossing singularities and $\dim Z \leq 4$ then $B_{(-Z)}Y \setminus Z$ is smooth in a neighborhood of Z .
- (6) $\omega_{B_{(-Z)}Y} \cong \pi^* \omega_Y \cong \pi^*(\omega_P \otimes L^{n-1})|_Y$ and the normal bundle of $\pi_*^{-1}Z$ in $B_{(-Z)}Y$ is $\omega_Z \otimes \omega_P^{-1} \otimes L^{1-n}$.

Proof. The claims (1–5) are étale local and, once they are established, (6) follows from the adjunction formula. Thus we can assume that $P = \mathbb{A}^N$. Next we write down étale local equations for Y and for $B_{(-Z)}Y$ and then read off their properties.

5 (Local computations). Let $Z \subset \mathbb{A}^N$ be a complete intersection of codimension n defined by $f_1 = \cdots = f_n = 0$. Let $Z \subset Y \subset \mathbb{A}^N$ be a general complete intersection of codimension $n-1$. It is thus defined by a system of equations

$$\begin{array}{ccccccc} h_{1,1}f_1 & + & \cdots & + & h_{1,n}f_n & = & 0 \\ \vdots & & & & \vdots & & \\ h_{n-1,1}f_1 & + & \cdots & + & h_{n-1,n}f_n & = & 0 \end{array} \quad (5.1)$$

Let $H = (h_{ij})$ be the matrix of the system and H_i the submatrix obtained by removing the i th column. Note that for h_{ij} general, the equations

$$(\text{rank } H < n-1) \quad \text{that is} \quad (\det H_1 = \cdots = \det H_n = 0) \quad (5.2)$$

define a codimension 2 subset of Z (6). If $f_n(\mathbf{x}) = 0$ then either $f_1(\mathbf{x}) = \cdots = f_{n-1}(\mathbf{x}) = 0$ or the system

$$\begin{array}{ccccccc} h_{1,1}(\mathbf{x}) \cdot y_1 & + & \cdots & + & h_{1,n-1}(\mathbf{x}) \cdot y_{n-1} & = & 0 \\ \vdots & & & & \vdots & & \\ h_{n-1,1}(\mathbf{x}) \cdot y_1 & + & \cdots & + & h_{n-1,n-1}(\mathbf{x}) \cdot y_{n-1} & = & 0 \end{array} \quad (5.3)$$

has a nontrivial solution $y_i = f_i(\mathbf{x})$, thus $\det H_n(\mathbf{x}) = 0$. We can do this for any j instead of n , hence we get that

$$(f_j = 0) = Z \cup (f_j = \det H_j = 0) \subset Y. \quad (5.4.j)$$

By our argument, this holds set-theoretically, but since Y is CM, (5.4.j) in fact holds scheme-theoretically. The formula (5.4.j) also suggests that computing

$$\pi : B_{(-Z)}Y := \text{Proj}_Y \sum_{m=0}^{\infty} \mathcal{O}_Y(mZ) \rightarrow Y$$

is the same as blowing up the ideal $(f_j, \det H_j)$ for any j :

$$B_{(-Z)}Y \cong B_{(f_j, \det H_j)}Y. \quad (5.5.j)$$

Again, because of the possible difference between the powers of the ideal $(f_j, \det H_j)$ and its symbolic powers, so far we only know that the S_2 -hull of $B_{(f_j, \det H_j)}Y$ is $B_{(-Z)}Y$. Next we show that $B_{(f_j, \det H_j)}Y$ is CM hence S_2 , thus (5.5.j) indeed holds.

Z is CM, hence (5.4.j) and (7) imply that $(f_j = \det H_j = 0)$ is CM and so is $(f_j = \det H_j = 0) \times \mathbb{P}^1$. In $Y \times \mathbb{P}^1$ the equation $(sf_j = t \det H_j)$ defines $B_{(f_j, \det H_j)}Y \cup ((f_j = \det H_j = 0) \times \mathbb{P}^1)$. Again using (7) we see that $B_{(f_j, \det H_j)}Y$ is CM.

The formula (5.5.j) shows that π is an isomorphism whenever $f_j \neq 0$ or $\det H_j \neq 0$. Letting j vary, the first set of these conditions define Z and the second set defines a codimension 2 subset of Z by (5.2).

For notational simplicity, let us compute the $j = n$ case. In $\mathbb{A}^N \times \mathbb{P}_{st}^1$ the blow-up satisfies the equations (5.1) and $sf_n = t \det H_n$. Multiplying the system (5.1) by the determinant-theoretic adjoint of H_n , we get the equations

$$\det H_n \cdot (f_1, \dots, f_{n-1})^{tr} + f_n \cdot H_n^{adj} \cdot (h_{1,n}, \dots, h_{n-1,n})^{tr} = 0. \quad (5.6)$$

Multiplying through by s , substituting $sf_n = t \det H_n$ and dividing by $\det H_n$ we get new equations for the blow-up:

$$s \cdot (f_1, \dots, f_{n-1})^{tr} + t \cdot H_n^{adj} \cdot (h_{1,n}, \dots, h_{n-1,n})^{tr} = 0 \quad (5.7)$$

These, together with $sf_n = t \det H_n$ show that $(t = 0) \subset B_{(f_n, \det H_n)}Y$ is isomorphic to Z . Furthermore, $H_n^{adj} \cdot (h_{1,n}, \dots, h_{n-1,n})^{tr}$ is the zero vector exactly where $\text{rank } H < n - 1$, proving (3).

Thus $U := (s \neq 0) \subset B_{(f_n, \det H_n)}Y$ is an open neighborhood of Z . In U , the equations (5.1) are consequences of (5.7) and $sf_n = t \det H_n$. Thus the n equations (5.7) and $sf_n = t \det H_n$ define U , hence it is a local complete intersection.

If Z has hypersurface singularities, then we can set f_1, \dots, f_{n-1} to be linear and independent. Thus the equations (5.7) can be used to eliminate variables. Possibly after shrinking U and choosing new étale coordinates, we end up with one equation

$$U := (f_n = t \det H_n) \subset \mathbb{A}_{(\mathbf{x}, t)}^{N-n+2}, \quad (5.8)$$

where H_n is a general $(n-1) \times (n-1)$ matrix of polynomials in the \mathbf{x} -variables.

If Z has normal crossing singularities then Z is slc, but if E is a divisor over Z with discrepancy ≤ 0 then $\text{center}_Z E$ is either a stratum of Z (if $\text{discrep}(E, Z) = -1$) or is a codimension 1 point in a stratum of Z (if $\text{discrep}(E, Z) = 0$) (cf. [KM98, 2.29]). Thus, by the precise inversion of adjunction [K⁺92, 17.3, 17.12], U is terminal near Z iff it is terminal at the codimension 1 points of the strata of Z . At these points, the equation (5.8) is

$$(x_1 \cdots x_m = t) \quad \text{or} \quad (x_1 \cdots x_m = tx_{m+1}) \subset \mathbb{A}_{(\mathbf{x}, t)}^{N-n+2}; \quad (5.9)$$

these are terminal singularities. For later purposes we also note the following.

Claim 5.10. Assume that Z has normal crossing singularities and L is ample.

- a) Every irreducible component of $\text{Sing } Z$ contains a point where, in suitable local analytic coordinates,

$$[Z \subset B_{(-Z)}Y] \cong [(t=0) \subset (x_1x_2 = tx_3)].$$

- b) $B_{(-Z)}Y$ is smooth at a general point of every stratum of Z . \square

If $\dim Z \leq 4$ then $\dim \text{Sing } Z \leq 3$ and by (6) we may assume that every point of $Z \subset B_{(-Z)}Y$ is described by a local equation $(x_1 \cdots x_m = t)$ or $(x_1 \cdots x_m = tx_{m+1})$ for some $m \leq 4$. This shows (5).

Starting with $\dim Z = 5$, we get singularities outside Z of the form

$$(x_1x_2 = t(x_3x_4 - x_5x_6)) \subset \mathbb{A}_{(\mathbf{x},t)}^7.$$

This is still of type cA . If $\dim Z = 6$ then we also get triple points of the form

$$(x_1x_2x_3 = t(x_4x_5 - x_6x_7)) \subset \mathbb{A}_{(\mathbf{x},t)}^8. \quad \square$$

6 (Determinantal varieties). We have used the following properties of determinantal varieties; see [BV88] for a general treatment.

Let V be a smooth, affine variety, $V_i \subset V$ a finite set of smooth, affine subvarieties and $\mathcal{L} \subset \mathcal{O}_X$ a finite dimensional base point free linear system. Let $H_{n,m} = (h_{ij})$ be an $n \times m$ matrix whose entries are general elements in \mathcal{L} . Then for every i

- (1) the singular set of $V_i \cap (\det H_{n,n} = 0)$ has codimension 4 in V_i and
- (2) the set $V_i \cap (\text{rank } H_{n,n-1} < n-1)$ has codimension 2 in V_i .

The following is a basic observation of liaison theory (cf. [Eis95, Sec.21.10]).

Lemma 7. *Let W be a Gorenstein scheme of pure dimension n that is a union of two of its closed subschemes W_1, W_2 of pure dimension n . If W_1 is CM then so is W_2 .*

Proof. Set $D := W_1 \cap W_2$. There is an exact sequence

$$0 \rightarrow \mathcal{O}_{W_2}(-D) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_{W_1} \rightarrow 0.$$

Here $\mathcal{O}_{W_2}(-D)$ is the largest subsheaf of \mathcal{O}_W whose support is in W_2 . This shows that $\mathcal{O}_{W_2}(-D)$ is CM. Similarly, we have

$$0 \rightarrow \mathcal{O}_{W_1}(-D) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_{W_2} \rightarrow 0.$$

Hom it to ω_W to get

$$0 \rightarrow \omega_{W_2} \rightarrow \omega_W \rightarrow \omega_{W_1}(D) \rightarrow 0.$$

Here again ω_{W_2} is the largest subsheaf of ω_W whose support is in W_2 . Since W is Gorenstein, $\mathcal{O}_W \cong \omega_W$ which implies that $\omega_{W_2} \cong \mathcal{O}_{W_2}(-D)$. Thus ω_{W_2} is CM and so is \mathcal{O}_{W_2} . \square

Let now Z be a projective, connected, local complete intersection scheme of pure dimension n and L an ample line bundle on Z . A large multiple of L embeds Z into $P := \mathbb{P}^N$ for some N . Applying (5) we get $Z \subset B_{(-Z)}Y$ where the normal bundle of Z is very negative. Thus Z can be contracted (analytically or as an algebraic space) and we get a singular point $(0 \in X)$. Its properties are summarized next.

Theorem 8. *Let Z be a projective, connected, local complete intersection scheme of pure dimension n and L an ample line bundle on Z . Then for $m \gg 1$ there are germs of normal singularities $(0 \in X = X(Z, L, m))$ with a partial resolution*

$$\begin{array}{ccc} Z & \subset & Y \\ \downarrow & & \downarrow \pi \\ 0 & \in & X \end{array} \quad \text{where } Y \setminus Z \cong X \setminus \{0\}$$

such that

- (1) Z is a Cartier divisor in Y ,
- (2) the normal bundle of Z in Y is $\omega_Z \otimes L^{-m}$,
- (3) if Z is snc then Y has only terminal singularities and (5.10.a-b) hold.
- (4) if $\dim Z \leq 4$ then $(0 \in X)$ is an isolated singular point. \square

Next we consider various properties of these $(0 \in X)$.

Proposition 9. *Let $0 \in X$ be a germ of a normal singularity with a partial resolution*

$$\begin{array}{ccc} Z & \subset & Y \\ \downarrow & & \downarrow \pi \\ 0 & \in & X \end{array} \quad \text{where } Y \setminus Z \cong X \setminus \{0\}.$$

Assume that Z is a reduced, slc, Cartier divisor with normal bundle L^{-1} where L is ample and $-K_Z$ is nef. Then:

- (1) $R^i \pi_* \mathcal{O}_Y \cong H^i(Z, \mathcal{O}_Z)$.
- (2) The restriction $\text{Pic}(Y) \rightarrow \text{Pic}(Z)$ is an isomorphism.
- (3) If Y is smooth then $\text{Cl}(X) = \text{Pic}(Y) / \langle [Z_i] : i \in I \rangle$ where the $\{Z_i : i \in I\}$ are the irreducible components of Z .
- (4) If $\mathcal{O}_Z(-K_Z) \cong L^r$ then
 - (a) K_X is Cartier and $(0 \in X)$ is lc,
 - (b) If $r > 0$ and Y has canonical singularities then $(0 \in X)$ is canonical.
 - (c) If $r > 1$, Y has canonical singularities and $Y \setminus Z$ has terminal singularities then $(0 \in X)$ is terminal.

Proof. We think of a germ as a small analytic neighborhood of $0 \in X$. To be precise, we assume that X is Stein and Z is a deformation retract of Y . This in particular implies that we have isomorphisms

$$H^*(Z, \mathbb{Z}) \cong H^*(Y, \mathbb{Z}) \quad \text{and} \quad \pi_1(Z) \cong \pi_1(Y). \quad (9.5)$$

Let $I_Z \subset \mathcal{O}_Y$ be the ideal sheaf of $Z \subset Y$. Then the completion of $R^i \pi_* \mathcal{O}_Y$ equals the inverse limit of the groups $H^i(Y, \mathcal{O}_Y / I_Z^m)$, hence it is enough to prove that the maps

$$H^i(Y, \mathcal{O}_Y / I_Z^{m+1}) \rightarrow H^i(Y, \mathcal{O}_Y / I_Z^m) \rightarrow \cdots \rightarrow H^i(Z, \mathcal{O}_Z)$$

are all isomorphism. For each of these we have an exact sequence

$$H^i(Z, L^m) \rightarrow H^i(Y, \mathcal{O}_Y / I_Z^{m+1}) \rightarrow H^i(Y, \mathcal{O}_Y / I_Z^m) \rightarrow H^{i+1}(Z, L^m)$$

and $H^i(Z, L^m) = 0$ by Kodaira vanishing, proving (1). (We need Kodaira vanishing in a slightly more general setting than usual [KM98, 2.4]. The normal crossing case is easy to derive by induction, or see [Kol95, 9.12, 12.10] for the general log canonical setting using [KK10].)

A similar argument proves (2) using the exact sequence

$$0 \rightarrow L^m \rightarrow (\mathcal{O}_Y / I_Z^{m+1})^* \rightarrow (\mathcal{O}_Y / I_Z^m)^* \rightarrow 1.$$

which gives

$$H^1(Z, L^m) \rightarrow \text{Pic}(\text{Spec}_Y \mathcal{O}_Y / I_Z^{m+1}) \rightarrow \text{Pic}(\text{Spec}_Y \mathcal{O}_Y / I_Z^m).$$

If Y is smooth then $\text{Cl}(Y) = \text{Pic}(Y)$ which implies (3).

Finally, by adjunction, $(K_Y - (r-1)Z)|_Z \sim 0$, hence $K_Y - (r-1)Z \sim 0$ by (2). Thus $K_X \sim \pi_*(K_Y - (r-1)Z)$ is Cartier, $K_Y \sim \pi^*K_X + (r-1)Z$ and the rest of (4) is clear. \square

Proposition 10. *Let Z be a connected, reduced, proper, Cartier divisor in a normal analytic space Y such that Z is a deformation retract of Y . For each irreducible component $Z_i \subset Z$ let $\gamma_i \subset Y \setminus Z$ be a small loop around Z_i .*

(1) *Assume that $\text{codim}_Z(Z \cap \text{Sing } Y) \geq 2$. Then the γ_i (and their conjugates) generate $\ker[\pi_1(Y \setminus Z) \rightarrow \pi_1(Y)]$.*

(2) *Assume in addition that Z is a normal crossing scheme and*
 (a) *every codimension 1 stratum of Z contains a node of Y and*
 (b) *every codimension 2 stratum of Z contains a smooth point of Y .*

Then $\ker[\pi_1(Y \setminus Z) \rightarrow \pi_1(Y)]$ is cyclic and is generated by any of the γ_i .

Proof. Let $\rho^0 : (Y \setminus Z)^\sim \rightarrow Y \setminus Z$ be an étale cover that is trivial on all the γ_i . This means that ρ^0 extends to an étale cover of $Y \setminus (Z \cap \text{Sing } Y)$. By a Lefschetz type theorem [Gro68, XIII.2.1] it then extends to an étale cover of Y . This proves (1).

Next let $\rho^0 : (Y \setminus Z)^\sim \rightarrow Y \setminus Z$ be an étale cover and $\rho : Y^\sim \rightarrow Y$ its extension to a normal, ramified cover of Y . Assume that there is at least one point $y \in \rho^{-1}(Z)$ where ρ is étale. We claim that ρ is then everywhere étale.

First we show that ρ is étale generically on every irreducible component of $\rho^{-1}(Z)$ and on every codimension 1 stratum of $\rho^{-1}(Z)$. Since $\rho^{-1}(Z)$ is connected in codimension 1, we only need to show that if \tilde{Z}_i, \tilde{Z}_j are two irreducible components of $\rho^{-1}(Z)$ such that they intersect in codimension 1 and ρ is generically étale along \tilde{Z}_i then it is also generically étale along \tilde{Z}_j and along $\tilde{Z}_i \cap \tilde{Z}_j$.

Here we use the existence of nodes (2.a). In local coordinates we have

$$(Z \subset Y) \cong (t=0) \subset (x_1x_2 = tx_3) \subset \mathbb{A}_{(x_1, \dots, x_n, t)}^{n+1}.$$

Note that $(x_1x_2 = tx_3) \setminus (t=0) \sim \mathbb{C}^* \times \mathbb{C}^{n-1}$, thus, in this neighborhood, the two loops γ_1 around $(t = x_1 = 0)$ and γ_2 around $(t = x_2 = 0)$ are homotopic. Thus any étale cover of $(x_1x_2 = tx_3) \setminus (t=0)$ that is unramified along $(t = x_1 = 0)$ is also unramified along $(t = x_2 = 0)$.

At a general point of a codimension 2 stratum of Z we use (2.b). Thus, in local coordinates we have

$$(Z \subset Y) \cong (t=0) \subset (x_1x_2x_3 = t) \subset \mathbb{A}_{(x_1, \dots, x_n, t)}^{n+1}$$

and a cover that is étale along Z is trivial. Finally, since Z is a hypersurface singularity, it is locally simply connected at codimension ≥ 3 points [Gro68, X.3.4]. Thus any (possibly ramified) cover of Z that is étale outside a subset of codimension ≥ 3 is everywhere étale. \square

Remark 11. Note that the seemingly artificial condition (10.2.a) is necessary, even if Y is smooth everywhere.

As an example, let S be a resolution of a rational surface singularity ($0 \in T$) with exceptional curve $E \subset S$. Take $Z := E \times \mathbb{P}^1 \hookrightarrow S \times \mathbb{P}^1 =: Y$.

Then Z is simply connected but $\pi_1(Y \setminus Z) \cong \pi_1(S \setminus E)$ is infinite, nonabelian as soon as T is not a quotient singularity.

The presence of nodes along the double locus of Z (4.10), which at first seemed to be a blemish of the construction, thus turned out to be of great advantage to us.

12. The kernel of $[\pi_1(Y \setminus Z) \rightarrow \pi_1(Y)]$ in (10) can be infinite cyclic; for instance this happens if Z is an abelian variety.

However, if $\pi_1(Z) = 1$ and $Z \subset Y$ is contractible then $\pi_1(Y \setminus Z)$ is finite. To see this note that since $\pi_1(Y \setminus Z)$ is abelian, it is enough to show that γ_i has finite order in $H_1(Y \setminus Z, \mathbb{Z})$. We can now repeatedly cut Y by hyperplanes until it becomes a smooth surface, hence a resolution of a normal surface singularity. The latter case is computed in [Mum61].

An especially simple situation is when $\omega_Z \cong L^r$ for some r . We can choose L to be non-divisible in $\text{Pic}(Z)$. Then the normal bundle of Z in Y is L^{r-m} , thus $\mathcal{O}_Y(-Z)$ is divisible by $(m-r)$ in $\text{Pic}(Y)/\langle [Z_i] : i \in I \rangle$. We can thus take a degree $(m-r)$ cyclic cover of $X \setminus \{0\}$ and replace $Z \subset Y$ with another diagram

$$\begin{array}{ccc} Z & \subset & \tilde{Y} \\ \downarrow & & \downarrow \pi \\ 0 & \in & \tilde{X} \end{array}$$

where $\tilde{Y} \setminus Z \cong \tilde{X} \setminus \{0\}$, the normal bundle of Z is L^{-1} and $\tilde{X} \setminus \{0\}$ is simply connected.

The singularities of \tilde{Y} are, however, a little worse than before. The argument after (5.8) shows that they are canonical, not terminal. At double points of Z , the original local equations $(x_1x_2 = t)$ or $(x_1x_2 = tx_3)$ become

$$(x_1x_2 = s^{m-r}) \quad \text{or} \quad (x_1x_2 = s^{m-r}x_r).$$

Construction of log canonical 3-fold singularities.

By (8–10), the following construction of reducible snc surfaces implies (2). The construction and the proof of its properties were clearly known to the authors of [FM83], though only some of it is there explicitly.

Proposition 13. *For every connected 2-manifold without boundary F there are (many) connected algebraic surfaces $Z = Z(F)$ with snc singularities such that*

- (1) $K_Z \sim 0$ if F is orientable and $2K_Z \sim 0$ if F is not orientable,
- (2) $h^i(Z, \mathcal{O}_Z) = h^i(F, \mathbb{C})$ and
- (3) $\pi_1(Z) \cong \pi_1(F)$.

We first describe the irreducible components of these surfaces and then explain how to glue them together.

14 (Rational surfaces with an anticanonical cycle). Fix $m \geq 1$ and let Z be a rational surface such that $-K_Z$ is linearly equivalent to a length m cycle of rational curves C_1, \dots, C_m . One can get such surfaces by starting with 3 lines in \mathbb{P}^2 , then repeatedly blowing up an intersection point and adding the exceptional curve to the collection of curves $\{C_j\}$.

Let L be an ample line bundle on Z and consider the sequence

$$0 \rightarrow L(-\sum C_j) \rightarrow L \rightarrow L|_{\sum C_j} \rightarrow 0.$$

Since $-\sum C_j \sim K_Z$, we see that $H^1(Z, L(-\sum C_j)) = 0$. Thus we have a surjection

$$H^0(Z, L) \twoheadrightarrow H^0(\sum C_j, L|_{\sum C_j}).$$

Since $\text{Pic}^0(\sum C_j) \cong \mathbb{G}_m$, $L|_{\sum C_j}$ has a section which has exactly one zero on each C_j (not counting multiplicities). Thus there is a divisor $A_L \in |L|$ such that $A_L \cap C_j$ is a single point for every j .

We would like to choose L such that $\deg L_{C_j}$ is independent of j . This is not always possible, but it can be arranged as follows.

Assume that the self intersections $(C_j \cdot C_j)$ are ≤ -2 for every j . (This can be achieved by blowing up points on the C_j if necessary.) Then, for any j , all the other curves form the resolution of a cyclic quotient singularity, and their intersection form is negative definite. Thus if H_j is any ample divisor on Z then there is an effective linear combination

$$H'_j := H_j + \sum_{i \neq j} a_i C_i$$

such that $(H'_j \cdot C_i) = 0$ for $i \neq j$ and $(H'_j \cdot C_j) > 0$. Set

$$H := \sum_j \frac{1}{(H'_j \cdot C_j)} \cdot H'_j.$$

H is an ample \mathbb{Q} -divisor that has degree 1 on every C_j . A suitable multiple gives the required line bundle L .

Let next $P \subset \mathbb{R}^2$ be a convex polygon with vertices v_1, \dots, v_m and sides $S_i = [v_i, v_{i+1}]$. We map P into the algebraic surface Z as follows.

We map the vertex v_i to the point $C_i \cap C_{i-1}$. We can think of $C_i \cong \mathbb{CP}^1$ as a sphere with $C_i \cap C_{i+1}$ as north pole, $C_i \cap C_{i-1}$ as south pole and the unique point $A_L \cap C_i$ as a point on the equator. We map the side S_i to a semicircle in $C_i \cong \mathbb{CP}^1 \sim S^2$ whose midpoint is $A_L \cap C_i$. Since $\pi_1(Z) = 1$, this mapping of the boundary of P extends to $\tau : P \rightarrow Z(\mathbb{C})$ (whose image could have self-intersections).

15 (Gluing rational surfaces with anticanonical cycles). Let F be a (connected) topological surface and T a triangulation of F .

Dual to T is a subdivision of F into polygons P_i such that at most 3 polygons meet at any point.

For each polygon P_i with sides $S_1^i, \dots, S_{m_i}^i$ (in this cyclic order) choose a rational surface Z_i with an anticanonical cycle of rational curves $C_1^i, \dots, C_{m_i}^i$ (in this cyclic order) and a map $P_i \hookrightarrow Z_i(\mathbb{C})$ as in (14).

If the sides S_j^i and S_k^l are identified on F by an isometry $\phi_{jk}^{il} : S_j^i \rightarrow S_k^l$, then there is a unique isomorphism $\Phi_{jk}^{il} : C_j^i \rightarrow C_k^l$ extending ϕ_{jk}^{il} .

These gluing data define a surface $Z = \cup_i Z_i$ and the maps τ_i glue to $\tau : F \rightarrow Z(\mathbb{C})$. Since only 3 polygons meet at any vertex, we get an snc surface. The curves H_i glue to an ample Cartier divisor H on Z .

We claim that τ induces an isomorphism $\pi_1(F) \rightarrow \pi_1(Z(\mathbb{C}))$. This is clear for the 1-skeleton where each 1-cell in F is replaced by a $\mathbb{CP}^1 \sim S^2$.

As we attach each polygon P_i to the 1-skeleton, we kill an element of the fundamental group corresponding to its boundary. On each rational surface Z_i with anticanonical cycle $\sum_j C_j^i$ we have a surjection

$$\pi_2(Z_i, \sum_j C_j^i) \twoheadrightarrow \pi_1(\sum_j C_j^i),$$

thus as we attach Z_i we kill the same element of the fundamental group. Thus $\pi_1(F) \cong \pi_1(Z(\mathbb{C}))$.

The statement about the homology groups is proved in [FM83, pp.26–27].

Construction of terminal 4-fold singularities.

Let W be a smooth Fano 3-fold of index 2. That is, W is smooth and there is an ample line bundle L such that $-K_W \sim L^2$. Then the cone

$$C(W, L) := \operatorname{Spec} \sum_{m \geq 0} H^0(W, L^m)$$

is a terminal singularity.

Such smooth Fano 3-folds have been classified, they give examples only up to embedding dimension 7.

As a generalization, one can try to look for Fano 3-folds of index 2 with normal crossing singularities. Thus the irreducible components of its normalization are normal crossing pairs (W_i, S_i) with an ample divisor H_i such that $-(K_{W_i} + S_i) \sim 2H_i$.

The first examples of such pairs that come to mind are $(\mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1)$ and $(\mathbb{Q}^3, \mathbb{P}^1 \times \mathbb{P}^1)$. These are all the examples where the underlying variety W is also Fano.

In (16) we construct an infinite sequence of index 2 log Fano pairs $(P_r, \mathbb{P}^1 \times \mathbb{P}^1)$ where P_r is a \mathbb{P}^2 -bundle over \mathbb{P}^1 .

Any 2 of these examples can be glued together to get infinitely many families of Fano 3-folds of index 2 with normal crossing singularities.

Then we apply (4) and (8–10) to conclude the proof of (1).

Example 16. For $r \geq 0$ set

$$\pi : P_r := \operatorname{Proj}_{\mathbb{P}^1}(E_r) \rightarrow \mathbb{P}^1 \quad \text{where} \quad E_r := \mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(r).$$

Write $\mathcal{O}_P(a, b) := \mathcal{O}_{P_r}(a) \times \pi^* \mathcal{O}_{\mathbb{P}^1}(b)$. Since $\pi_* \mathcal{O}_P(a, b) = S^a E_r \otimes \mathcal{O}_{\mathbb{P}^1}(b)$, we see that $\mathcal{O}_P(a, b)$ is ample iff $a > 0$ and $b > 0$. Let $S_r \subset P_r$ be the surface corresponding to the unique section of $\mathcal{O}_P(1, -r)$. Note that $K_{P_r} \sim \mathcal{O}_P(-3, r-2)$, $S_r \cong \operatorname{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and

$$-(K_{P_r} + S_r) \sim \mathcal{O}_P(2, 2) \quad \text{is ample on } P_r.$$

For any $a, b \geq 0$ there are natural isomorphisms

$$H^0(P_r, \mathcal{O}_P(a, b)) = H^0(\mathbb{P}^1, S^a E_r \otimes \mathcal{O}_{\mathbb{P}^1}(b))$$

and $S^a E_r \otimes \mathcal{O}_{\mathbb{P}^1}(b)$ naturally decomposes as the sum of line bundles of the form $\mathcal{O}_{\mathbb{P}^1}(cr+b)$ where $0 \leq c \leq a$. This makes it easy to compute the spaces $H^0(P_r, \mathcal{O}_P(a, b))$ and to show the following

(1) For $a, b \geq 0$ the restriction maps

$$H^0(P_r, \mathcal{O}_P(a, b)) \rightarrow H^0(S_r, \mathcal{O}_S(a, b)) \quad \text{are surjective.}$$

(2) For $a_i, b_i \geq 0$ the multiplication maps

$$H^0(P_r, \mathcal{O}_P(a_1, b_1)) \otimes H^0(P_r, \mathcal{O}_P(a_2, b_2)) \rightarrow H^0(P_r, \mathcal{O}_P(a_1 + a_2, b_1 + b_2))$$

are surjective.

17 (Fano 3-folds of index 2). We consider in detail two series of examples.

1. Fix $r \geq 0$ and let Z_r be obtained from (P_r, S_r) and (\mathbb{P}^3, S_0) by an isomorphism $S_r \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong S_0$. Then ω_{Z_r} is ample and isomorphic to L_r^{-2} where L_r is ample.

One can check that $h^0(Z_r, L_r) = r + 6$ and the algebra $\sum_{m \geq 0} H^0(Z_r, L_r^m)$ is generated by $H^0(Z_r, L_r)$.

2. Fix $r, s \geq 0$ and let Z_{rs} be obtained from (P_r, S_r) and (P_s, S_s) by an isomorphism $S_r \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong S_s$. Then $\omega_{Z_{rs}}$ is ample and isomorphic to L_{rs}^{-2} where L_{rs} is

ample. Using (16.1–2) we compute that $h^0(Z_{rs}, L_{rs}) = r + s + 8$ and the algebra $\sum_{m \geq 0} H^0(Z_{rs}, L_{rs}^m)$ is generated by $H^0(Z_{rs}, L_{rs})$.

For $r \in \{0, 1\}$, the (Z_r, L_r) series should give degenerations of smooth Fano 3-folds. The simplest is $r = 0$. Take $\mathbb{P}^1 \times \mathbb{P}^2$ and embed it into \mathbb{P}^5 by $\mathcal{O}(1, 1)$. Under this embedding, $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^2$ becomes a quadric; it is thus contained in a 3-plane H^3 . The union of $\mathbb{P}^1 \times \mathbb{P}^2$ and of the 3-plane gives Z_0 . It is a $(2, 2)$ complete intersection.

The construction of X_r can also be realized by taking the cone over Z_r and then deforming the (reducible) cone by taking high-order perturbations of the two quadratic defining equations.

Putting these together with (8) and (12) we get the following.

Proposition 18. *Let (Z, L) be any of the pairs from (17). Then there are 4-dimensional, normal, isolated singularities $0 \in X$ with a partial resolution $\pi : (Z \subset Y) \rightarrow (0 \in X)$ such that*

- (1) Y has only canonical singularities of type cA ,
- (2) $Z \subset Y$ is a Cartier divisor and its normal bundle is L^{-1} ,
- (3) $0 \in X$ is terminal, K_X is Cartier, $X \setminus \{0\}$ is simply connected and
- (4) the embedding dimension of $(0 \in X)$ is $h^0(Z, L)$. □

It remains to compute the class group of X . We apply (9.3), which needs a resolution of singularities of Y . Our set-up is simple enough that this can be done explicitly.

19 (Explicit resolution). Let Y be a 4-fold and $Z \subset Y$ a Cartier divisor. Assume that Z is the union of 2 smooth components $Z = Z_1 \cup Z_2$ and along the intersection $S := Z_1 \cap Z_2$ in suitable local analytic coordinates $[Z_1 \cup Z_2 \subset Y]$ is isomorphic to

$$\begin{aligned} [(x_1 = s = 0) \cup (x_1 = s = 0) \subset (x_1 x_2 = s^m)] &\subset \mathbb{A}^5 \quad \text{or} \\ [(x_1 = s = 0) \cup (x_1 = s = 0) \subset (x_1 x_2 = s^m x_3)] &\subset \mathbb{A}^5. \end{aligned}$$

Let $C \subset S$ be the curve defined locally by $(x_1 = x_2 = x_3 = s = 0)$.

We can resolve the singularities by iterating the following steps.

- (1) If $m \geq 3$, we blow up S . We get 2 exceptional divisors, both are \mathbb{P}^1 -bundles over S with 2 disjoint sections. In the above local coordinates the equation changes to $(x_1 x_2 = s^{m-2})$ or $(x_1 x_2 = s^{m-2} x_3)$.
- (2) If $m = 2$, we blow up S ; the resulting 4-fold is smooth. We get 1 exceptional divisor, which is a conic bundle over S with 2 disjoint sections. It is isomorphic to a \mathbb{P}^1 -bundle over S blown up along C (contained in one of the sections).
- (3) If $m = 1$, we blow up the component, call it N , that intersects Z_1 . (This component is Z_2 iff there are no previous blow-ups.) The birational transform of Z_1 is isomorphic to Z_1 and the birational transform of N is isomorphic to N blown-up along C .

Thus at the end we have a chain of smooth 3-folds

$$E_0 := Z_1, E_1, \dots, E_{m-1}, E_m := Z_2$$

such that

- (4) the only intersections are $S_i := E_i \cap E_{i+1} \cong S$ for $0 \leq i < m$.
- (5) E_2, \dots, E_{m-1} are \mathbb{P}^1 -bundles over S with 2 disjoint sections

- (6) E_1 is a \mathbb{P}^1 -bundle over S blown up along C if $m \geq 2$ and Z_2 blown up along C if $m = 1$.

By taking the cohomology of the exact sequence

$$0 \rightarrow \mathbb{Z}_{\cup_i E_i} \rightarrow \sum_{i=0}^m \mathbb{Z}_{E_i} \rightarrow \sum_{i=0}^{m-1} \mathbb{Z}_{S_i} \rightarrow 0$$

we get an exact sequence

$$\sum_{i=0}^{m-1} H^1(S_i, \mathbb{Z}) \rightarrow H^2(\cup_i E_i, \mathbb{Z}) \rightarrow \sum_{i=0}^m H^2(E_i, \mathbb{Z}) \rightarrow \sum_{i=0}^{m-1} H^2(S_i, \mathbb{Z}).$$

Assume now that

$$H^1(S, \mathbb{Z}) = 0 \quad \text{and} \quad H^2(Z_2, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z}) \quad \text{is surjective.} \quad (19.7)$$

Setting $h^i(\cdot) := \text{rank } H^i(\cdot, \mathbb{Z})$ we get that

$$\begin{aligned} h^2(\cup_i E_i) &= \sum_{i=0}^m h^2(E_i) - \sum_{i=0}^{m-1} h^2(S_i) \\ &= h^2(Z_1) + h^2(Z_2) - h^2(S) + h^2(C) + (m-1). \end{aligned}$$

In (17) $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and C is connected, thus the formula becomes

$$h^2(\cup_i E_i) = h^2(Z_1) + h^2(Z_2) + (m-2).$$

Therefore, by (9.3), the class group of X , obtained by contracting $Z \subset Y$ to a point, satisfies

$$\begin{aligned} \text{rank Cl}(X) &\leq h^2(Z_1) + h^2(Z_2) + (m-2) - (m+1) \\ &= h^2(Z_1) + h^2(Z_2) - 3. \end{aligned} \quad (19.8)$$

Thus we see that for the series $(0 \in X_r)$ we get $\text{Cl}(X_r) = 0$ but for the series $(0 \in X_{rs})$ we get $\text{Cl}(X_{rs}) \cong \mathbb{Z}$.

Remark 20. The series X_{rs} can also be constructed as follows. Set

$$Y_{rs} := \text{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(r) + \mathcal{O}_{\mathbb{P}^1}(s)).$$

This visibly contains both P_r, P_s as divisors and $K_{Y_{rs}} + P_r + P_s \sim \mathcal{O}_{Y_{rs}}(-2, -2)$. We can now take the affine cone $C_a(Y_{rs}, \mathcal{O}_{Y_{rs}}(1, 1))$ and inside it the cone $C_a(P_r + P_s)$. Perturbing the equation of the cone $C_a(P_r + P_s)$ as in [KM98, 2.43] we get our varieties X_{rs} .

Since $\text{Cl}(X_{rs}) = \mathbb{Z}$, the singularities X_{rs} have a small modification; now we can see this explicitly. The cone $C_a(Y_{rs})$ has a small resolution determined by the pencil of the 4-planes that are the cones over the fibers of $Y_{rs} \rightarrow \mathbb{P}^1$.

By contrast, I believe that one can not realize the series Z_r as hypersurfaces in a smooth variety.

Remark 21. Although these examples show that terminal 4-fold singularities do not form an “essentially bounded family” in the most naive sense, it should not be considered a final answer.

There are several ways to “simplify” a given terminal singularity $(0 \in X)$.

First, if $X \setminus \{0\}$ is not simply connected, one can pass to the universal cover; see (26). Since finite group actions on a given singularity are frequently not too hard to understand, it makes sense to concentrate on those singularities $(0 \in X)$ for which $X \setminus \{0\}$ is simply connected.

A harder to use reduction step is the following. If there is a Weil divisor $D \subset X$ that is not \mathbb{Q} -Cartier, then there is a (unique) small modification $g : X_D \rightarrow X$ such that $g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample. This X_D also has terminal singularities. Even for 3-folds this reduction is quite subtle since the existence of such a D depends on

the coefficients of the equations. (For instance, if $X_f := (x_1x_2 + f(x_3, x_4) = 0) \subset \mathbb{A}^4$ then $\text{Cl}(X_f) = 0$ iff $f(x_3, x_4)$ is irreducible (as a power series); see [Kol91, 2.2.7].)

Nonetheless this suggests that the most basic terminal singularities ($0 \in X$) of dimension n are those that satisfy both $\pi_1(X \setminus \{0\}) = 1$ and $\text{Cl}(X) = 0$.

Finally, one might argue that any collection of examples constructed in a simple uniform way forms an “essentially bounded family.” In essence, I have just replaced the old recipe “take a cone over a smooth Fano 3-fold of index 2” with a newer recipe “take a cone over an snc Fano 3-fold of index 2 and deform it.” The complete list of snc Fano 3-folds of index 2 (and also many higher dimensional cases) is in [Fuj11]. Thus my examples do form an “essentially bounded family.”

Questions.

These examples raise several questions.

Question 22. Is there a more conceptual way to construct all these examples?

Comments: A quite general approach could be the following.

Let $0 \in X \subset \mathbb{A}^N$ be a subscheme and assume that the deformations of all singularities of $X \setminus \{0\}$ are unobstructed. (For instance, X could be a cone over a projective variety whose singularities have unobstructed deformations.) Then all the obstructions to deforming X are supported at the origin, hence the obstruction space for X should be finite dimensional. On the other hand, if $0 \in X$ is not an isolated singularity, then the deformation space itself should be infinite dimensional.

We can thus expect that deformations of X , even those deformations that induce a flat deformation of the tangent cone at $0 \in X$, give pretty much a “general” deformation at the singularities of $X \setminus \{0\}$.

A technical difficulty in carrying this out is that infinite dimensional deformation spaces are difficult to handle. It is also not clear what to expect over $X \setminus \{0\}$.

As an example, assume that $X \subset \mathbb{A}^N$ given by a determinantal condition ($\text{rank } H_{r,r+1} < r$) as in (6). Then every deformation of X is also determinantal. The singular set is then given by the condition ($\text{rank } H_{r,r+1} < r - 1$), which is a subset of codimension 6. We aim to keep the origin still singular, thus, if $\dim X \geq 7$, we can not smooth $X \setminus \{0\}$.

As we saw in (4), we can still expect for a general deformation to have very nice singularities outside the origin. I do not know any definite general results.

Question 23. Let Z be a projective snc scheme with $K_Z \sim 0$. Is there any restriction on $\pi_1(Z)$?

Comments: We can try to follow the construction on (15) but already in dimension 3 this seems quite difficult.

Assume that we have a polyhedron P with boundary $F \sim S^2$. As in (15), we can use F and its triangulation to build a surface Z such that $K_Z \sim 0$. The next step would be to find a 3-fold X that contains Z as a divisor such that $K_X + Z \sim 0$. It is not clear that this is always possible. Even if for each polyhedron P such a 3-fold $X(P)$ exists, gluing them together is probably quite a bit more subtle than for surfaces.

Here is a quite interesting simple case.

Let P be the dodecahedron. For each face we choose a degree 5 del Pezzo surface; it contains a length 5 chain of -1 -curves that form an anticanonical cycle. We can glue these together to get a surface Z with 12 irreducible components such that

$K_Z \sim 0$. I do not know how to construct a rationally connected 3-fold X containing Z as a divisor such that $K_X + Z \sim 0$.

This example would be quite interesting since many hyperbolic 3-manifolds admit a tiling with copies of the dodecahedron.

By an observation of [Sim10], for each finitely presented group Γ there is a seminormal scheme $Z(\Gamma)$ such that $\pi_1(Z(\Gamma)) \cong \Gamma$.

The following variant was explained to me by M. Kapovich.

Start with a finite simplicial complex C whose fundamental group is Γ . For each k -simplex $c \in C$ we take $Z(c) := \mathbb{CP}^k$. Use the incidence relation in C to glue the spaces $Z(c)$ together (using linear embeddings). The result is a singular projective scheme $Z(\Gamma)$ whose fundamental group is Γ .

If C is a topological manifold, then $K_{Z(\Gamma)} \sim 0$ and the singularities of $Z(\Gamma)$ are simple normal crossing in codimension 1 but more complicated in codimension ≥ 2 . In codimension 2 we get degenerate cusps, and their deformation theory is quite subtle [GHK11]. I do not know if these examples can be realized as exceptional sets of partial resolutions of log canonical singularities.

Kapovich also outlined an argument of how to modify this construction to obtain a projective variety $Y(\Gamma)$ with simple normal crossing singularities whose fundamental group is Γ . However, the canonical class of $Y(\Gamma)$ is not trivial.

Question 24. Let $(0 \in X)$ be the germ of a log canonical singularity and $g : Y \rightarrow X$ a resolution. Is there any restriction on $\pi_1(Y)$?

Question 25. Let (X, D) be an lc pair with $K_X + D \sim_{\mathbb{Q}} 0$. What are the possible groups $\pi_1(D)$? Is $\pi_1(D)$ a birational invariant of the pair (X, D) ? What about $\pi_1([\Delta])$ if (X, Δ) is lc and $K_X + \Delta \sim_{\mathbb{Q}} 0$?

Comments: Let (X, D) be an lc pair with $K_X + D \sim_{\mathbb{Q}} 0$. If $D \neq 0$ then X is uniruled and if D is rationally chain connected then, by looking at the MRC fibration we conclude that X is rationally connected. This implies that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i$. By looking at the exact sequence

$$0 \rightarrow \omega_X \cong \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

we conclude that $H^i(D, \mathcal{O}_D) = 0$ for $0 < i < \dim D$. It is natural to hope that $\pi_1(D)$ should be finite.

Question 26. Let $(0 \in X)$ be the germ of a log terminal singularity. Is $\pi_1(X \setminus \{0\})$ finite?

Comments: This is probably the most basic of the above questions. It is closely related to the following old problem:

Let (X, Δ) be a dlt Fano variety. Is the fundamental group of the smooth locus X^{ns} finite? The answer is yes in dimension 2 but even that case needs work [GZ95, FKL93, KM99].

Acknowledgments. This paper was written while I visited RIMS, Kyoto University. I thank S. Mori and S. Mukai for the invitation and their hospitality. I am grateful to K. Fujita, M. Kapovich, S. Mori, C. Simpson and Ch. Xu for many comments and corrections. Partial financial support was provided by the NSF under grant number DMS-0758275.

REFERENCES

- [BR10] Gavin Brown and Miles Reid, *Anyone knows these guys?*, <http://dl.dropbox.com/u/10909533/anyoneknows.pdf>, 2010.
- [BV88] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988. MR 953963 (89i:13001)
- [Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)
- [Elk81] R. Elkik, *Rationalité des singularités canoniques*, Inv. Math. **64** (1981), 1–6.
- [FKL93] Akira Fujiki, Ryoichi Kobayashi, and Steven Lu, *On the fundamental group of certain open normal surfaces*, Saitama Math. J. **11** (1993), 15–20. MR 1259272 (94m:32042)
- [FM83] Robert Friedman and David R. Morrison (eds.), *The birational geometry of degenerations*, Progr. Math., vol. 29, Birkhäuser Boston, Mass., 1983. MR 690262 (84g:14032)
- [Fuj11] Kento Fujita, (*in preparation*), 2011.
- [GHK11] M. Gross, P. Hacking, and S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, arXiv1106.4977G, 2011.
- [Gro68] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, North-Holland Publishing Co., Amsterdam, 1968, Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2. MR 0476737 (57 #16294)
- [GZ95] R. V. Gurjar and D.-Q. Zhang, π_1 of smooth points of a log del Pezzo surface is finite. II, J. Math. Sci. Univ. Tokyo **2** (1995), no. 1, 165–196. MR 1348027 (96i:14015)
- [K⁺92] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Soc. Math. France, Astérisque vol. 211, 1992.
- [KK10] János Kollár and Sándor J. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813. MR 2629988
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)
- [KM99] Seán Keel and James McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999), no. 669, viii+153. MR 1610249 (99m:14068)
- [Kol91] János Kollár, *Flips, flops, minimal models, etc*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 113–199. MR 1144527 (93b:14059)
- [Kol93] ———, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math. **113** (1993), no. 1, 177–215. MR 1223229 (94m:14018)
- [Kol95] ———, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1995. MR 1341589 (96i:14016)
- [Mum61] David Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 9, 5–22. MR 0153682 (27 #3643)
- [Rei80] Miles Reid, *Canonical 3-folds*, Journées de Géométrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 273–310. MR 605348 (82i:14025)
- [Rei87] ———, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414. MR 927963 (89b:14016)
- [Rei94] ———, *Nonnormal del Pezzo surfaces*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 5, 695–727. MR 1311389 (96a:14042)
- [Sim10] Carlos Simpson, *Local systems on proper algebraic V-manifolds*, arXiv1010.3363, 2010.
- [Tak03] Shigeharu Takayama, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math. **14** (2003), no. 8, 825–836. MR 2013147 (2004m:14023)

Princeton University, Princeton NJ 08544-1000

kollar@math.princeton.edu